Three-Dimensional Geometry

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1 Distance Formula

The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2 Section Formula

If a point R divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in the ratio $m : n$, then:

$$R = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$

Applies to internal and external division.

3 Direction Cosines and Ratios

Let a line make angles α, β, γ with x, y, z axes. Then,

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma$$

are called direction cosines.

If a, b, c are proportional to l, m, n, then (a, b, c) are direction ratios.

- Two vectors are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
- Two vectors are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

4 Equation of a Line

(i) Vector Form – Point and Direction Vector

A line passing through a point $\vec{a}$ and parallel to vector $\vec{b}$:

$$\vec{r} = \vec{a} + \lambda\vec{b}, \quad \lambda \in \mathbb{R}$$

(ii) Cartesian Form – Point and Direction Ratios

Through point (x_1, y_1, z_1) and direction ratios l, m, n:

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

(iii) Line Through Two Points

Let $A(\vec{a})$, $B(\vec{b})$ be points. Then the line through A and B is:

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$$

5 Angle Between Two Lines

If lines have direction vectors $\vec{b}_1$ and $\vec{b}_2$, then the angle θ is:

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$$

6 Shortest Distance Between Two Skew Lines

If lines are:

$$\vec{r}_1 = \vec{a}_1 + \lambda\vec{b}_1, \quad \vec{r}_2 = \vec{a}_2 + \mu\vec{b}_2$$

Then shortest distance:

$$d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$$

7 Distance from a Point to a Line

Given line $\vec{r} = \vec{a} + \lambda \vec{b}$ and point $\vec{p}$:

$$d = \frac{|\vec{b} \times (\vec{p} - \vec{a})|}{|\vec{b}|}$$

8 Equation of a Plane

(i) Through Three Points

If A, B, C are points with position vectors $\vec{a}, \vec{b}, \vec{c}$:

$$\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a}) + \mu(\vec{c} - \vec{a})$$

(ii) Intercept Form

If plane intercepts axes at $(a, 0, 0), (0, b, 0), (0, 0, c)$:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

9 Angle Between Two Planes

If normals of planes are $\vec{n}_1$ and $\vec{n}_2$, then:

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}$$

10 Planes Parallel and Intersecting

(i) Parallel Plane

If $ax + by + cz + d_1 = 0$, then a parallel plane is $ax + by + cz + d_2 = 0$.

(ii) Plane Through Line of Intersection

If two planes are:

$$\vec{r} \cdot \vec{n}_1 = d_1, \quad \vec{r} \cdot \vec{n}_2 = d_2$$

Then the plane through their intersection line:

$$\vec{r} \cdot (\vec{n}_1 + \lambda \vec{n}_2) = d_1 + \lambda d_2$$

11 Distance Between a Point and a Plane

For point (x_1, y_1, z_1) and plane $ax + by + cz + d = 0$:

$$\text{Distance} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

12 Distance Between Parallel Planes

If planes are:

$$ax + by + cz + d_1 = 0, \quad ax + by + cz + d_2 = 0$$

Then distance:

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

13 Bisector Planes Between Two Planes

Given two planes:

$$a_1x + b_1y + c_1z + d_1 = 0, \quad a_2x + b_2y + c_2z + d_2 = 0$$

Then bisector planes:

$$\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

14 Angle Between a Line and a Plane

If line has direction vector $\vec{b}$ and plane has normal $\vec{n}$, then:

$$\sin\theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}||\vec{n}|}$$

Special Conditions

- Line parallel to plane: $\vec{b} \cdot \vec{n} = 0$
- Line lies in plane: $\vec{a} \cdot \vec{n} = d$ and $\vec{b} \cdot \vec{n} = 0$