Binomial Theorem

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1 Introduction

A **binomial expression** consists of exactly two terms. For example: $x+y$, $a-b$, $\frac{x}{y}+1$.

If an expression has more than two terms, it is called a **multinomial**.

2 Binomial Theorem (for Positive Integer Index)

For any natural number n:

$$(x+a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} a^r$$

Key Notes

- $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ are called **binomial coefficients**.
- Total number of terms $= n+1$.
- Sum of exponents of x and a in each term $= n$.
- If $\binom{n}{x} = \binom{n}{y}$, then either $x = y$ or $x + y = n$.

General Term

The $(r+1)^{\text{th}}$ term in the expansion is:

$$T_{r+1} = \binom{n}{r} x^{n-r} a^r$$

Middle Term(s)

- If n is even, the middle term is the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term.
- If n is odd, the middle terms are the $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ terms.

Maximum Binomial Coefficient

- For even n: Maximum value is at $r = \frac{n}{2}$.
- For odd n: Maximum values at $r = \frac{n-1}{2}$ and $r = \frac{n+1}{2}$.

3 Identities and Useful Results

Expansions

$$(x+a)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} a^r$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$$

Special Values

- Put $x = 1$: $\Rightarrow 2^n = \sum_{r=0}^{n} \binom{n}{r}$
- Differentiating gives:

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}$$

- Integrating gives:

$$\int (1+x)^n dx = \sum_{r=0}^{n} \binom{n}{r} \frac{x^{r+1}}{r+1} + C$$

4 Numerically Greatest Term

In the expansion $(a+x)^n$, to find the numerically greatest term:

$$\frac{T_{r+1}}{T_r} = \frac{n-r+1}{r} \cdot \frac{x}{a}$$

Solve for:

$$\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow r \leq \frac{nx}{a+x}$$

Let:

$$r = \left\lfloor \frac{nx}{a+x} \right\rfloor$$

Then:

- If result is not an integer: Greatest term $= T_{r+1}$
- If result is an integer: Both T_r and T_{r+1} are numerically greatest

5 Binomial Expansion for Any Index

The expansion of $(1+x)^n$ for any real or rational n (not necessarily a positive integer) is given by:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Conditions

- Valid only for $|x| < 1$
- The number of terms is infinite (non-terminating series)

General Term

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r$$

Important Expansions

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots = \sum_{r=0}^{\infty} (-1)^r x^r, \quad |x| < 1$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots = \sum_{r=0}^{\infty} x^r, \quad |x| < 1$$