Matrices and Determinants

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1 Matrix: Definition and Types

A **matrix** is a rectangular arrangement of elements in rows and columns. A matrix with m rows and n columns is said to be of **order** $m \times n$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Common Types of Matrices

- **Zero Matrix:** All elements are zero.
- **Square Matrix:** Number of rows = number of columns ($m = n$).
- **Diagonal Matrix:** A square matrix where non-diagonal elements are zero.
- **Identity Matrix:** A diagonal matrix with 1s on the main diagonal.

2 Matrix Operations

Addition and Subtraction

Matrices of the same order can be added or subtracted element-wise.

Matrix Multiplication

If A is of order $m \times n$ and B is of order $n \times p$, then the product AB is defined and is of order $m \times p$.

Transpose

The transpose of matrix A (denoted A^T) is obtained by interchanging rows and columns.

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{then } A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

3 Special Matrices

- **Symmetric Matrix:** $A^T = A$
- **Skew-Symmetric Matrix:** $A^T = -A$
- **Orthogonal Matrix:** $AA^T = A^T A = I$

4 Minors and Cofactors

Minor

The minor of an element a_{ij} in a matrix is the determinant of the submatrix formed by deleting the i-th row and j-th column.

Cofactor

The cofactor C_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

5 Rank of a Matrix

The **rank** of a matrix A, denoted by $\rho(A)$, is the maximum order of a non-zero minor in A.

Properties

- Rank remains unchanged under elementary row/column operations.
- Rank of a zero matrix is 0.
- A skew-symmetric matrix cannot have rank 1.
- $\rho(A) = \rho(A^T)$

6 Solving Linear Equations using Matrices

A system of equations:

$$AX = B$$

has a unique solution if $|A| \neq 0$, given by:

$$X = A^{-1}B$$

7 Determinants

Minor of a Determinant

Minor of element a_{ij} is the determinant obtained by removing its i-th row and j-th column.

Cofactor

$$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

8 Properties of Determinants

1. Swapping rows/columns changes the sign.
2. Identical rows or columns $\rightarrow$ determinant is 0.
3. Multiplying a row/column by scalar k multiplies the determinant by k.
4. If two rows/columns are proportional $\rightarrow$ determinant is 0.
5. Transposing a determinant doesn't change its value.

9 Cramer's Rule

2-Variable System

Given:

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

Solution:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$

where:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

3-Variable System

Given:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D}$$

D is the determinant of coefficients; D_x, D_y, and D_z are obtained by replacing respective columns with the constants.

Conclusion

This chapter introduces matrices and determinants, crucial for linear algebra and systems of equations. Mastery of operations, special matrix forms, and solution techniques is essential for problem-solving in advanced mathematics.