# Permutation and Combination

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## 1 Fundamental Principles of Counting

### Multiplication Principle

If a task can be performed in $m$ ways, and another independent task can be done in $n$ ways, then the total number of ways to do both tasks in sequence is $m \times n$.

### Addition Principle

If one task can be done in $m$ ways and another task in $n$ ways, and only one of them is to be done, then the total number of ways is $m + n$.

## 2 Permutations

A **permutation** is an arrangement of objects in a specific order. The order of arrangement is crucial in permutations.

### Number of Permutations

The number of ways to arrange $r$ objects out of $n$ distinct objects is:

$$^nP_r = \frac{n!}{(n-r)!}$$

where $0 \leq r \leq n$.

#### Special Cases

- $^nP_0 = 1$
- $^nP_1 = n$
- $^nP_n = n!$

Also,

$$^nP_r = n \cdot {}^{n-1}P_{r-1}$$

### Permutations with Repetition

If repetition is allowed, then the number of permutations of $n$ objects taken $r$ at a time is:

$$n^r$$

### Permutations of Non-distinct Items

If among $n$ objects, some are identical—say $x$, $y$, and $z$ are alike of three different kinds—then the number of unique permutations is:

$$\frac{n!}{x! \cdot y! \cdot z!}$$

## 3 Circular Permutations

### Case 1: All items in a circle

If $n$ distinct objects are arranged in a circle, then the number of circular permutations is:

$$(n-1)!$$

### Case 2: Selecting $r$ items for circular arrangement

- If clockwise and anti-clockwise arrangements are considered different: $\frac{^nP_r}{r}$
- If both directions are considered the same: $^nP_r$

- If both directions are considered the same: $\frac{\quad}{2r}$

# 4 Combinations

A **combination** refers to selection or grouping where the order of selection is irrelevant.

### Number of Combinations

The number of ways to choose $r$ elements from a set of $n$ distinct elements is:

$$^nC_r = \frac{n!}{r!(n-r)!}$$

**Properties**

- $^nC_0 = {}^nC_n = 1$
- $^nC_1 = n$
- $^nC_r = {}^nC_{n-r}$

### Maximum Value of $^nC_r$

- If $n$ is even: maximum value $= {}^nC_{n/2}$
- If $n$ is odd: maximum value $= {}^nC_{(n-1)/2} = {}^nC_{(n+1)/2}$

# 5 Selection from Objects

### From Distinct Objects

The number of non-empty subsets of a set of $n$ distinct objects:

$$2^n - 1 = {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n$$

### From Identical and Distinct Objects

If there are $a_1, a_2, \ldots, a_n$ identical items of $n$ kinds and $k$ distinct objects, then the number of ways of selecting at least one item:

$$(a_1 + 1)(a_2 + 1) \ldots (a_n + 1) \cdot 2^k - 1$$

# 6 Division of Objects into Groups

### From Distinct Objects

To divide $m + n + p$ objects into 3 groups of sizes $m$, $n$, and $p$, the number of groupings is:

$$\frac{(m+n+p)!}{m! \cdot n! \cdot p!}$$

### Equal Division into Distinct Groups

To divide $mn$ distinct objects equally into $m$ distinct groups of size $n$:

$$\frac{(mn)!}{(n!)^m}$$

### Division of Identical Objects

To divide $n$ identical objects into $r$ groups:

- **If empty groups are allowed:** $^{n+r-1}C_{r-1}$
- **If empty groups are not allowed:** $^{n-1}C_{r-1}$

# 7 Arrangements Within Groups

To distribute and arrange $n$ distinct objects into $r$ groups:

- **If empty groups are allowed:** $n! \cdot {}^{n+r-1}C_{r-1}$
- **If not allowed:** $n! \cdot {}^{n-1}C_{r-1}$