Complex Numbers and Quadratic Equations

Content curated by Chandra Shekhar

1 Complex Numbers

A **complex number** is an expression of the form $z = a+ib$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$.

- $\text{Re}(z) = a$ is the real part.
- $\text{Im}(z) = b$ is the imaginary part.
- If $a = 0$, z is purely imaginary.
- If $b = 0$, z is purely real.

Polar (Trigonometric) Form

Let $z = a + ib = r(\cos\theta + i\sin\theta)$, where

$$r = |z| = \sqrt{a^2 + b^2}, \quad \theta = \arg(z)$$

This is known as the polar/trigonometric form of a complex number. The general form includes multiple values of the argument:

$$z = r\left[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)\right], \quad n \in \mathbb{Z}$$

2 Properties of Complex Conjugates

If $z = a + ib$, then $\bar{z} = a - ib$ is the conjugate of z. Properties include:

- $z\bar{z} = |z|^2$
- $z = \bar{z} \iff z$ is real
- $z = -\bar{z} \iff z$ is purely imaginary
- $\text{Re}(z) = \frac{z+\bar{z}}{2}$
- $\text{Im}(z) = \frac{z-\bar{z}}{2i}$

3 Modulus Properties

- $|z| \geq 0$ with equality iff $z = 0$
- $|z| = |\bar{z}| = |-z|$
- $|z_1 z_2| = |z_1||z_2|$
- $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangle inequality)

4 Argument of a Complex Number

- $\arg(z)$ is the angle θ made by z with the positive x-axis.
- Quadrant-based examples:
  - $z = 1 + i \Rightarrow \arg z = \frac{\pi}{4}$
  - $z = -1 + i \Rightarrow \arg z = \frac{3\pi}{4}$
  - $z = -1 - i \Rightarrow \arg z = -\frac{3\pi}{4}$
  - $z = 1 - i \Rightarrow \arg z = -\frac{\pi}{4}$

Argument Identities

- $\arg(z_1 z_2) = \arg z_1 + \arg z_2 + 2k\pi$
- $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$
- $\arg(z^n) = n \cdot \arg(z) + 2k\pi$
- $\arg(z) = -\arg(\bar{z})$

5 De Moivre's Theorem

For any real θ and integer n,

For any real θ and integer n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Inverse Forms

- $(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$
- $(\cos\theta + i\sin\theta)^{-n} = \cos(n\theta) - i\sin(n\theta)$

6 Cube Roots of Unity

The solutions of $z^3 = 1$ are:

$$1, \quad \omega = \frac{-1+\sqrt{3}i}{2}, \quad \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

- $\omega^3 = 1$
- $1 + \omega + \omega^2 = 0$

7 Quadratic Equations

A **quadratic equation** is of the form:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Roots

The roots α, β are given by:

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Nature of Roots

- Real and distinct if $D > 0$
- Real and equal if $D = 0$
- Complex (conjugate pair) if $D < 0$

Relations Between Roots

$$\alpha + \beta = -\frac{b}{a},$$
$$\alpha\beta = \frac{c}{a}$$

8 Graph of a Quadratic Expression

For $f(x) = ax^2 + bx + c$, the vertex form is:

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}, \quad D = b^2 - 4ac$$

- Vertex: $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$
- Opens upwards if $a > 0$, downwards if $a < 0$

9 Quadratic Inequations

Inequalities of the form $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c \leq 0$ are quadratic inequations. They are solved based on the sign of the expression between and beyond the roots.

10 Polynomial Roots: Key Theorems

Factor Theorem

If $f(\alpha) = 0$, then $(x - \alpha)$ is a factor of $f(x)$.

Multiplicity

If a root α appears m times, then:

$$f(x) = (x - \alpha)^m \cdot g(x)$$

Odd-Degree Polynomial

Every polynomial of odd degree has at least one real root.

Rolle's Theorem (for Polynomials)

If $f(a) = f(b)$ and $f(x)$ is continuous and differentiable in $[a, b]$, then $\exists c \in (a, b)$ such that $f'(c) = 0$.