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4 Conic Sections Overview

A conic is defined as the locus of a point whose distance from a fixed point (focus) and a fixed line (directrix) is constant.

- $e = 1$: Parabola
- $e < 1$: Ellipse
- $e > 1$: Hyperbola
- $e = 0$: Circle

5 Parabola

Standard Forms of Parabola

Equation	Opens	Axis	Vertex
$y^2 = 4ax$	Right	Horizontal	$(0, 0)$
$y^2 = -4ax$	Left	Horizontal	$(0, 0)$
$x^2 = 4ay$	Upward	Vertical	$(0, 0)$
$x^2 = -4ay$	Downward	Vertical	$(0, 0)$
$(y - k)^2 = 4a(x - h)$	Right	Horizontal	(h, k)
$(x - h)^2 = 4a(y - k)$	Upward	Vertical	(h, k)

Important Elements

Element	Expression
Focus	$(h + a, k)$ for $(y - k)^2 = 4a(x - h)$
Directrix	$x = h - a$ for $(y - k)^2 = 4a(x - h)$
Axis	Line passing through vertex and focus
Latus Rectum (Length)	$4a$
Ends of Latus Rectum	$(h + a, k \pm 2a)$
Focal Distance of a Point (x, y)	$\|y^2 - 4ax\|/\sqrt{1 + (dy/dx)^2}$

Parametric Coordinates

Parabola	Parametric Point
$y^2 = 4ax$	$(at^2, 2at)$
$x^2 = 4ay$	$(2at, at^2)$

Tangent and Normal

Tangent at Point $(at^2, 2at)$

$$y = tx + a/t$$

Tangent Equation (General Point (x_1, y_1))

$$yy_1 = 2a(x + x_1) \quad (\text{for } y^2 = 4ax)$$

Normal at Point $(at^2, 2at)$

$$y = -tx + 2at + at^3$$

Slope of Tangent at (x, y)

$$\frac{dy}{dx} = \frac{2a}{y} \quad \text{for } y^2 = 4ax$$

Chord and Related Formulas

- Length of chord through point (x_1, y_1) with slope m:

$$\text{Length} = \frac{y_1^2 - 4ax_1}{am}$$

- Midpoint of chord in parabola $y^2 = 4ax$ lies on the line:

$$T = S_1 \Rightarrow yy_1 = 2a(x + x_1)$$

Key Concepts to Crack JEE Problems on Parabola

1. **Always identify the parabola form.** Convert general form to standard form by completing the square.
2. **Use parametric form** $(at^2, 2at)$ to represent points on the parabola for locus and geometric problems.
3. **Tangent/normal problems:** Derive slopes using parametric coordinates or differentiation.
4. **Latus rectum-based questions:** Remember endpoints, length is always $4a$.
5. **Focus-directrix definition:** Use definition $PF = e \cdot PD$ with $e = 1$.
6. **Intersection with line:** Substitute line in parabola and solve quadratic to get points.
7. **Minimum/maximum distance:** Often reduced to geometric interpretation (use symmetry).
8. **Area under parabola:** Use definite integration if bounded by lines/axes.
9. **Chord with given midpoint:** Use the property that midpoint satisfies $T = S_1$.
10. **Reflections and focal properties:** Parabolas reflect rays from focus parallel to axis.

6 Ellipse

Standard Forms of Ellipse

Equation	Type	Major Axis	Centre
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Horizontal	Along x-axis	$(0, 0)$
$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$	Vertical	Along y-axis	$(0, 0)$
$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	Shifted	Depends on $a > b$	(h, k)

Key Elements of Ellipse

Element	Expression
Major Axis Length	$2a$
Minor Axis Length	$2b$
Foci (Horizontal)	$(\pm c, 0)$ where $c = \sqrt{a^2 - b^2}$
Foci (Vertical)	$(0, \pm c)$ where $c = \sqrt{a^2 - b^2}$
Eccentricity	$e = \frac{c}{a}$
Latus Rectum	$\frac{2b^2}{a}$
Directrices (Horizontal)	$x = \pm\frac{a}{e}$
Directrices (Vertical)	$y = \pm\frac{a}{e}$

Parametric Coordinates

Form	Parametric Point
Horizontal Ellipse	$(a\cos\theta, b\sin\theta)$
Vertical Ellipse	$(b\cos\theta, a\sin\theta)$

Tangent and Normal

Tangent at Parametric Point $(a\cos\theta, b\sin\theta)$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

Tangent at Point (x_1, y_1)

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Normal at Parametric Point $(a\cos\theta, b\sin\theta)$

$$\frac{x}{\cos\theta a} - \frac{y}{\sin\theta b} = a\sin\theta - b\cos\theta \quad \text{(derived form, not always needed)}$$

Slope of Tangent at (x, y)

$$\frac{dy}{dx} = -\frac{b^2x}{a^2y}$$

Chords and Geometry

- **Length of Latus Rectum:** $\frac{2b^2}{a}$
- **Equation of Chord with Midpoint** (x_1, y_1):

$$T = S_1 \Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

- **Focal Property:** Sum of distances from any point on ellipse to both foci is constant:

$$PF_1 + PF_2 = 2a$$

Key Concepts to Solve JEE Problems on Ellipse

1. **Identify orientation:** Based on larger denominator ($a^2 > b^2$) $\to$ Horizontal/Vertical.
2. **Parametric form** $(a\cos\theta, b\sin\theta)$ is powerful for locus, tangents, and area problems.
3. **Use focal definition:** $PF_1 + PF_2 = 2a$ often simplifies geometric problems.
4. **Tangent and normal problems:** Use point-form or parametric-form equations.
5. **Chord midpoint formula:** Use $T = S_1$ when given midpoint of chord.
6. **Maximum/Minimum distance:** Use symmetry or calculus — max from center, min from axis.
7. **Area under ellipse segment:** Integration or area formula: πab for full ellipse.
8. **Focal chord properties:** Special chords through focus; product of slopes = constant.
9. **Eccentricity clues:** Ellipse if $e < 1$, use it to find directrix, latus rectum, etc.
10. **Locus questions:** Express point using parameters or focal property and eliminate.

7 Hyperbola

Standard Forms of Hyperbola

Equation	Type	Transverse Axis	Centre
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Horizontal	Along x-axis	$(0,0)$
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	Vertical	Along y-axis	$(0,0)$
$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Shifted	Depends on axis	(h,k)

Key Elements of Hyperbola

Element	Expression
Transverse Axis Length	$2a$
Conjugate Axis Length	$2b$
Foci (Horizontal)	$(\pm c, 0)$ where $c = \sqrt{a^2 + b^2}$
Foci (Vertical)	$(0, \pm c)$ where $c = \sqrt{a^2 + b^2}$
Eccentricity	$e = \frac{c}{a}$
Latus Rectum	$\frac{2b^2}{a}$
Directrices (Horizontal)	$x = \pm\frac{a}{e}$
Directrices (Vertical)	$y = \pm\frac{a}{e}$
Asymptotes (Horizontal)	$y = \pm\frac{b}{a}x$
Asymptotes (Vertical)	$y = \pm\frac{a}{b}x$

Parametric Coordinates

Form	Parametric Point
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$(a\sec\theta, b\tan\theta)$
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$	$(b\tan\theta, a\sec\theta)$

Tangent and Normal

Tangent at Parametric Point $(a\sec\theta, b\tan\theta)$

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

Tangent at Point (x_1, y_1)

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Normal at Parametric Point $(a\sec\theta, b\tan\theta)$

$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 \quad \text{(or derived form)}$$

Slope of Tangent at (x, y)

$$\frac{dy}{dx} = \frac{b^2x}{a^2y} \quad \left(\text{for } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right)$$

Chords and Properties

- **Focal Property:**

$$|PF_1 - PF_2| = 2a \quad \text{(constant)}$$

- **Equation of Chord with Midpoint** (x_1, y_1):

$$T = S_1 \Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

- **Length of Latus Rectum:** $2b^2/a$
- **Product of slopes of conjugate diameters:** $-b^2/a^2$

Key Concepts to Solve JEE Problems on Hyperbola

1. **Distinguish from ellipse:** Look for the minus sign. Ellipse has '+'; hyperbola has ''.
2. **Asymptotes:** Useful in limits, approximation, and infinite area questions.
3. **Use parametric form** $(a\sec\theta, b\tan\theta)$ in locus and geometric problems.
4. **Focal definition:** Difference of distances from foci = constant $(2a)$.
5. **Tangent/normal problems:** Use both point and parametric forms.
6. **Chord with given midpoint:** Use $T = S_1$ formula — works for ellipse and hyperbola.
7. **Rectangular hyperbola:** Special case with $a = b$, equation is $xy = c^2$.
8. **Transformations:** Rotate rectangular hyperbola $xy = c^2$ to $X^2 - Y^2 = 2c^2$ using

8. **Transformations:** Rotate rectangular hyperbola $xy = c^2$ to $X^2 - Y^2 = 2c^2$ using $45°$ rotation.

9. **Use eccentricity:** $e = \sqrt{1 + \frac{b^2}{a^2}} > 1$

10. **Reflections/focus-directrix:** Reflective property is inverse of ellipse.

Rectangular Hyperbola

A special case where $a = b$, so asymptotes are perpendicular. Its equation is:

$$xy = c^2$$