Mathematical Concepts

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1 Introduction to Mathematical Induction

The term **induction** refers to a logical method where a general result is inferred from specific examples. It begins with concrete observations and progresses toward a universal conclusion.

While an observed pattern may seem true based on initial cases, its validity must be established through rigorous proof—or falsified via a counter-example.

2 What is a Proposition?

A **proposition** (or statement) is a declarative sentence that is either true or false.

When a proposition depends on a natural number n, we denote it as $P(n)$.

Example:

$$P(n): \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

3 Principle of Mathematical Induction

To prove a statement $P(n)$ is true for all $n \in \mathbb{N}$, the following steps are used:

Step-I: **Base Case (Verification):** Prove that $P(n)$ holds true for the initial value, typically $n = 1$ (or $n = k$, depending on context).

Step-II: **Inductive Hypothesis:** Assume $P(k)$ is true for some arbitrary $k \geq 1$.

Step-III: **Inductive Step:** Prove that $P(k+1)$ is also true using the assumption that $P(k)$ holds.

If both the base case and inductive step are valid, then by the principle of induction, $P(n)$ is true for all natural numbers $n \geq 1$.

4 First Principle of Mathematical Induction

The **first principle** of induction proves a statement $P(n)$ to be true for all $n \in \mathbb{N}$ (or $n \geq k$), using:

- $P(k)$ is true for some initial value $k \in \mathbb{N}$.
- $P(m) \Rightarrow P(m+1)$ for all $m \geq k$.

Then $P(n)$ is true for all $n \geq k$.

5 Procedure to Apply Induction

To prove a proposition $P(n)$ using the first principle:

1. **Base Step:** Verify that the statement is true for the starting value, say $n = k$.
2. **Assumption:** Assume the statement holds for $n = m$, i.e., $P(m)$ is true.
3. **Proof Step:** Using this assumption, show that $P(m+1)$ is also true.

6 Example

Let us consider the proposition:

$$P(n): \quad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Step I: Check for $n = 1$:

$$LHS = 1^2 = 1, \quad RHS = \frac{1(1+1)(2\cdot 1+1)}{6} = 1$$

Step II: Assume true for $n = k$:

$$1^2 + 2^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

Step III: Prove for $n = k + 1$:

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{(k+1)\left[k(2k+1) + 6(k+1)\right]}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Thus, by mathematical induction, the formula holds for all $n \in \mathbb{N}$.