Sets, Relations, and Functions

Content curated by Chandra Shekhar

1 Set Theory: Fundamental Concepts

A **set** is a well-defined collection of distinct objects grouped based on a common rule. The objects in a set are called **elements**. If an element x belongs to a set A, we write $x \in A$; otherwise, $x \notin A$.

Examples of Sets:

- Set of vowels: $\{a, e, i, o, u\}$
- Set of natural numbers less than 10: $\{1, 2, 3, \ldots, 9\}$

Types of Sets

- **Finite Set:** Contains countable elements. E.g., $\{2, 4, 6, 8\}$
- **Infinite Set:** Has uncountable elements. E.g., $\mathbb{N} = \{1, 2, 3, \ldots\}$
- **Empty Set:** Contains no elements; denoted by $\emptyset$ or $\{\}$
- **Singleton Set:** Contains exactly one element, e.g., $\{7\}$
- **Universal Set:** Set containing all elements under consideration, usually denoted by U
- **Power Set:** The set of all subsets of a set A, denoted by $\mathcal{P}(A)$.
  If $A = \{1, 2\}$, then $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$
  A set with n elements has 2^n subsets.

Set Operations

- **Union:** $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- **Intersection:** $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- **Difference:** $A - B = \{x : x \in A \text{ and } x \notin B\}$
- **Complement:** $A^c = \{x \in U : x \notin A\}$
- **Disjoint Sets:** A and B are disjoint if $A \cap B = \emptyset$

De Morgan's Laws

$$(A \cup B)^c = A^c \cap B^c$$
$$(A \cap B)^c = A^c \cup B^c$$

Cardinality Rules

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$n(A) = n(A \cap B) + n(A - B)$$
$$n(B) = n(A \cap B) + n(B - A)$$

2 Cartesian Product

Given two sets A and B, the Cartesian product $A \times B$ is:

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

If $A = \{1, 2\}$ and $B = \{x, y\}$, then

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

3 Relations

A **relation** from A to B is a subset of $A \times B$. If $(x, y) \in R$, we say x is related to y and write $x \, R \, y$.

- **Domain:** Set of all first elements in R
- **Range:** Set of all second elements in R

4 Functions

A **function** $f : A \to B$ maps each element of A to a unique element in B.

- A: Domain
- B: Codomain
- $f(x)$: Image of x

Example: $f(x) = x^2$ is a function from $\mathbb{R} \to \mathbb{R}$

Types of Functions

- **One-One (Injective):** Each element of the domain maps to a distinct element in codomain
- **Many-One:** Two or more domain elements map to the same codomain element
- **Onto (Surjective):** Every element of the codomain has a pre-image
- **Into:** Some codomain elements are not mapped
- **Bijective:** Both one-one and onto

Algebra of Functions

Given $f : D_1 \to \mathbb{R}$ and $g : D_2 \to \mathbb{R}$:

$$\begin{aligned}(f+g)(x) &= f(x) + g(x) \\ (f-g)(x) &= f(x) - g(x) \\ (f \cdot g)(x) &= f(x) \cdot g(x) \\ \left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}, \quad \text{where } g(x) \neq 0\end{aligned}$$

Domain $D = D_1 \cap D_2$ (with necessary restrictions for division)

5 Special Functions

- **Even Function:** $f(x) = f(-x)$
- **Odd Function:** $f(-x) = -f(x)$
- **Periodic Function:** $f(x + T) = f(x)$ for some T

Common Functions and Their Domain/Range

Function $f(x)$	Domain	Range
$\sin x$	$\mathbb{R}$	$[-1, 1]$
$\cos x$	$\mathbb{R}$	$[-1, 1]$
a^x $(a > 1)$	$\mathbb{R}$	$(0, \infty)$
$\log_a x$ $(a > 0, a \neq 1)$	$(0, \infty)$	$\mathbb{R}$
$[x]$ (floor function)	$\mathbb{R}$	$\mathbb{Z}$
$\lvert x \rvert$	$\mathbb{R}$	$[0, \infty)$
$\{x\}$ (fractional part)	$\mathbb{R}$	$[0, 1)$

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